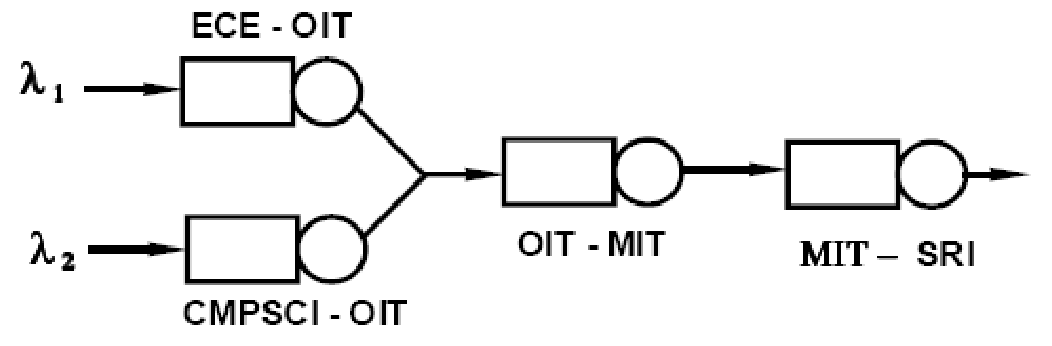
CDA6530: Performance Models of Computers and Networks

Chapter 7: Basic Queuing Networks

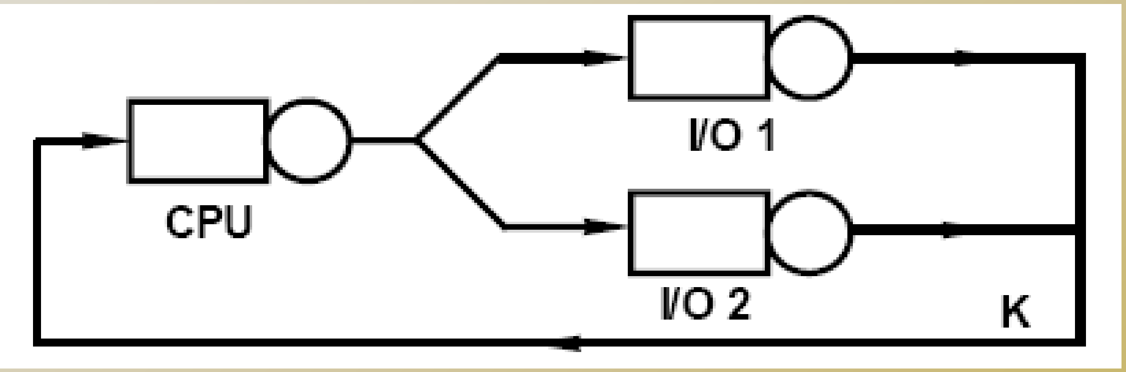
Open Queuing Network

- Jobs arrive from external sources, circulate, and eventually depart



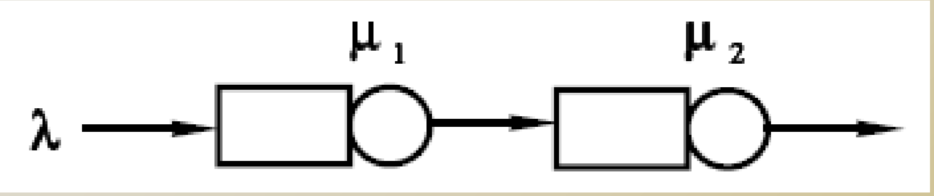
Closed Queuing Network

- Fixed population of *K* jobs circulate continuously and never leave
  - Previous machine-repairman problem



Feed-Forward QNs

- Consider two queue tandem system

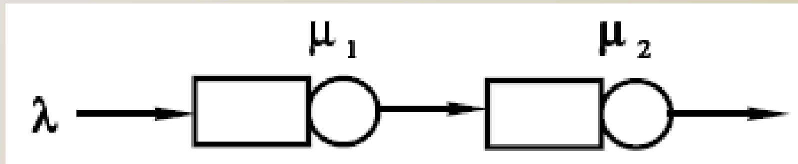


- Q: how to model?
  - System is a continuous-time Markov chain (CTMC)
  - State $(N_1(t), N_2(t))$, assume to be stable
  - $\pi(i,j) = P(N_1=i, N_2=j)$
  - Draw the state transition diagram
    - But what is the arrival process to the second queue?

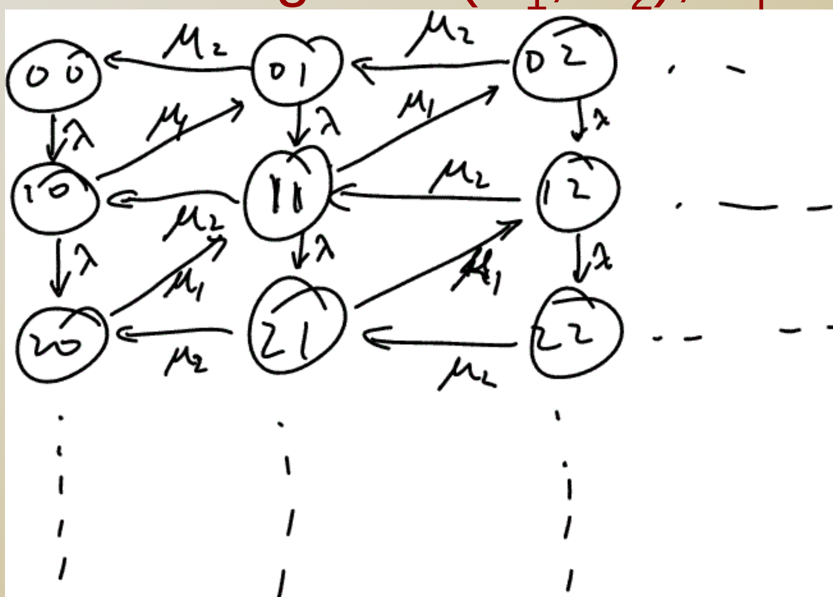
Poisson in $\Rightarrow$ Poisson out

- **Burke's Theorem:** Departure process of *M/M/*1 queue is Poisson with rate *λ* independent of arrival process.

- **Poisson process addition, thinning**
  - Two *independent* Poisson arrival processes adding together is still a Poisson ($\lambda=\lambda_1+\lambda_2$) Why?
  - For a Poisson arrival process, if each customer lefts with prob. p, the remaining arrival process is still a Poisson ($\lambda = \lambda_1 \cdot$ p)



- State transition diagram: (N_1, N_2), $N_i = 0, 1, 2, \cdots$



$$\pi(i,j) = (1-\rho_1)\rho_1^i(1-\rho_2)\rho_2^j \quad i,j \geq 0$$

$$\rho_i = \lambda/\mu_i$$

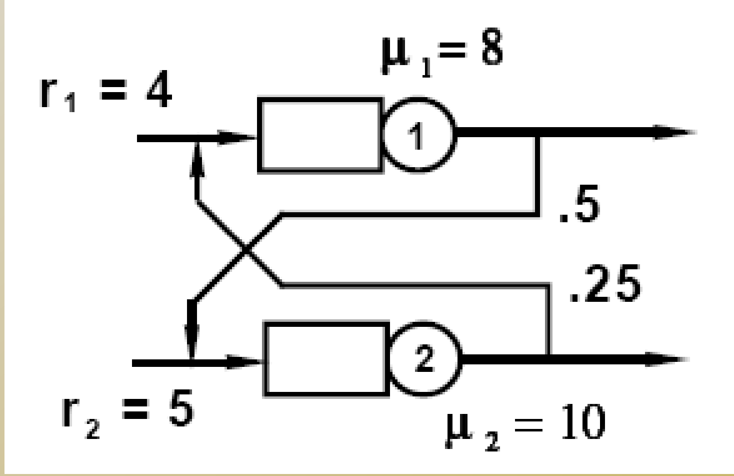
- For a k queue tandem system with Poisson arrival and expo. service time
- Jackson's theorem:

$$P(N_1 = n_1, N_2 = n_2, \ldots, N_k = n_k) = \prod_{i=1}^{k}(1-\rho_i)\rho_i^{n_i},$$

- Above formula is true when there are feedbacks among different queues
  - Each queue behaves as M/M/1 queue in isolation

Example

- λ_i: arrival rate at queue i



$$\lambda_1 = 4 + \lambda_2/4$$
$$\lambda_2 = 5 + \lambda_1/2$$

Why?

$$\Rightarrow \lambda_1 = 6,\ \lambda_2 = 8$$

$$\pi(n_1, n_2) = \frac{1}{4}\left(\frac{3}{4}\right)^{n_1} \frac{1}{5}\left(\frac{4}{5}\right)^{n_2}$$

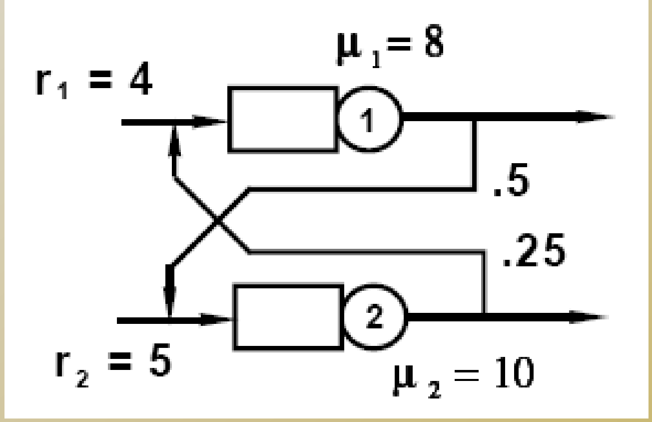
In M/M/1: $E[N] = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$

$$E[N] = \sum_{i=1}^{2} E[N_i] = \sum_{i=1}^{2} \lambda_i/(\mu_i - \lambda_i) = 3 + 4 = 7$$

$$E[T] = E[N]/(r_1 + r_2) = 7/9 \text{ time units}$$

Why?

- $T^{(i)}$: response time for a job enters queue i



$$E[T^{(1)}] = 1/(\mu_1 - \lambda_1) + E[T^{(2)}]/2$$
$$E[T^{(2)}] = 1/(\mu_2 - \lambda_2) + E[T^{(1)}]/4$$

Why?

In M/M/1: $E[T] = \frac{1}{\mu - \lambda}$

Extension

- results hold when nodes are multiple server nodes (*M/M/c*), infinite server nodes finite buffer nodes (*M/M/c/K*) (careful about interpretation of results), PS (process sharing) single server with arbitrary service time distr.

Closed QNs

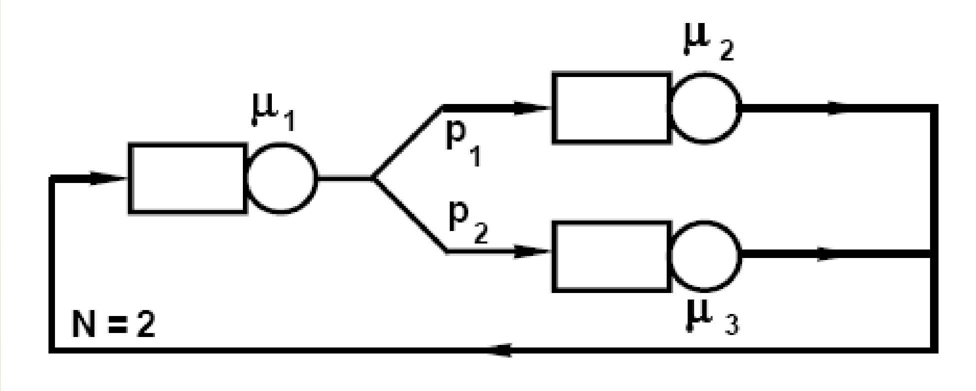
- Fixed population of *N* jobs circulating among *M* queues.
  - single server at each queue, exponential service times, mean $1/\mu_i$ for queue *i*
  - routing probabilities $p_{i,j}$, $1 \le i, j \le M$
  - visit ratios, $\{v_i\}$. If $v_1 = 1$, then v_i is mean number of visits to queue *i* between visits to queue 1

$$v_i = \sum_{j=1}^{M} v_j p_{j,i} \quad i = 2, \ldots M$$

  - γ_i: throughput of queue *i*,

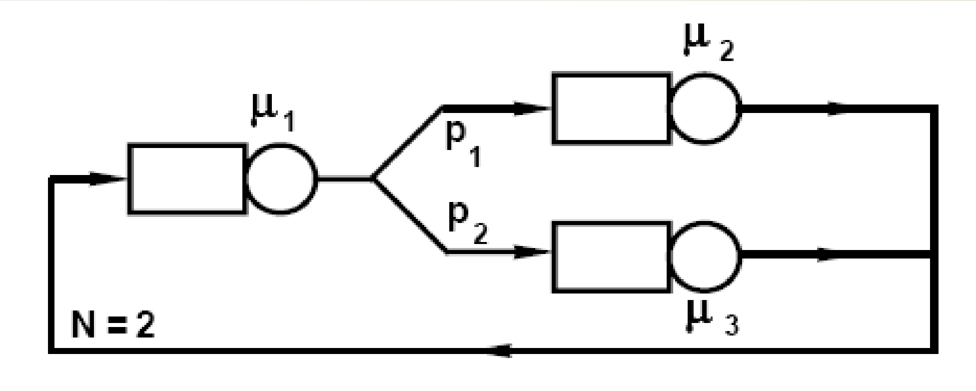
$$\gamma_i / \gamma_j = v_i / v_j, \quad 1 \le i, j \le M$$

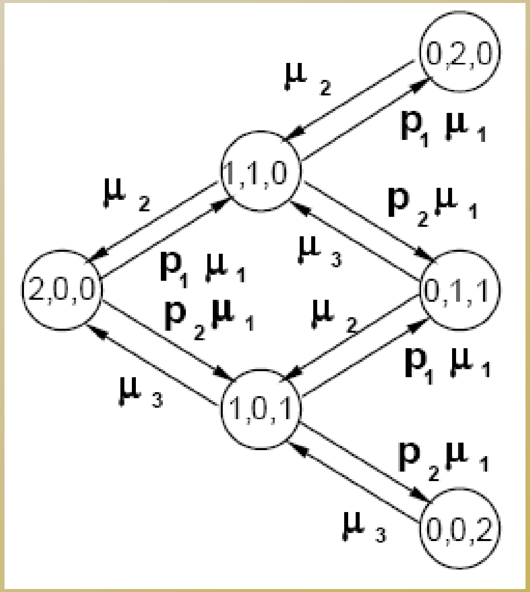
Example



- Open QN has infinite no. of states
- Closed QN is simpler

- How to define states?
  - No. of jobs in each queue


μ 1
μ 2
μ 3
p 1
p 2
N = 2


0,2,0
1,1,0
2,0,0
0,1,1
1,0,1
0,0,2
μ 2
p1 μ 1
p 2 μ 1
μ 3

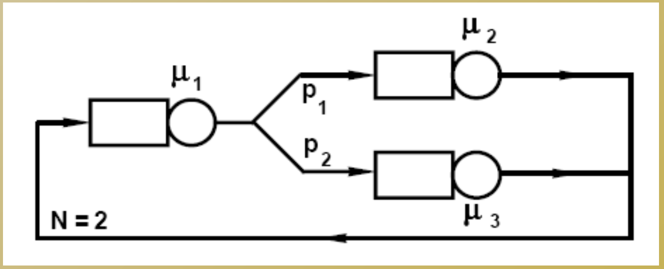
Steady State Solution

- **Theorem (Gordon and Newell)**

$$\pi(\vec{n}) = \frac{1}{G(N)} \prod_{i=1}^{M} \left(\frac{v_i}{\mu_i}\right)^{n_i} \qquad \vec{n} \geq \vec{0}; \sum_{i=1}^{M} n_i = N$$

where $\vec{n} = (n_1, \ldots, n_M)$, and $G(N)$ is a constant chosen so that $\sum \pi(\vec{n}) = 1$.

- For previous example when p1=0.75 , v_i?



$$v_1 = 1,\ v_2 = 3/4,\ v_3 = 1/4$$